Additional Problems 15.3

compute the first order partial derivatives.

1. $z = \ln(x^2+y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}, \quad \frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$$

2. $z = \tan(x/y)$

$$\frac{\partial z}{\partial x} = \frac{\sec^2(x/y)}{y}, \quad \frac{\partial z}{\partial y} = -\frac{x\sec^2(x/y)}{y^2}$$

3. Use the linear approximation to estimate the percentage change in volume of a right-circular cone of radius $r = 40$cm if the height is increased from 40 to 41 cm.

The volume of a cone is: $V = \frac{1}{3}\pi r^2 h$.

Then, $\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$ and

$$V(h+\Delta h, r) - V(h, r) \approx \frac{\partial V}{\partial h}(h, r) \cdot \Delta h$$

So, $\Delta V \approx \frac{1}{3}\pi (40)^2 \cdot 1$.

So, percent change:

$$\frac{\Delta V}{V} \approx \frac{1/3\,\pi (40)^2}{1/3\,\pi (40)^2 \cdot 40} = 1/40 \approx \boxed{0.025}.$$

Show that the following functions are harmonic:

4. $u(x,y) = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0. \checkmark$$

5. $u(x,y) = \ln(x^2+y^2)$.

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} = \frac{-2y^2+2x^2}{(x^2+y^2)^2}$$

So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-2x^2+2y^2-2y^2+2x^2}{(x^2+y^2)^2} = 0$ ✓

6. Show that if $u(x,y)$ is harmonic, then the partial derivatives $\frac{\partial u}{\partial x}$ and $\partial u/\partial y$ are harmonic.

$u(x,y)$ harmonic $\Rightarrow u_{xx} + u_{yy} = 0$.

We compute Δu_x,

$$\Delta u_x = \underbrace{u_{xxx} + u_{xyy}}_{}$$ thm 1 on p. 777, u smooth $\Rightarrow$ continuity

$$= \frac{\partial}{\partial x}(u_{xx} + u_{yy}) = 0$$

Similar for u_y.

1. By theorem 1 on p. 784, we have: If $f(x,y)$ is locally linear at (a,b), then its tangent plane is given by the equation

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$f(x,y) = x^2y + xy^3$, $(1,2)$. $\Rightarrow f(1,2) = 2+8 = 10$

$f_x(x,y) = 2xy + y^3$ $\Rightarrow$ $f_x(1,2) = 2(1)(2) + 2^3 = 4+8 = 12$

$f_y(x,y) = x^2 + 3xy^2$ $\quad$ $f_y(1,2) = 1^2 + 3(1)(2)^2 = 1+12 = 13$

$\Rightarrow z = 10 + 12(x-1) + 13(y-2)$

$\Rightarrow z = 10 + 12x - 12 + 13y - 26$

$\Rightarrow \boxed{z = 12x + 13y - 28}$

2. $\sin(uw)$ at $(\pi/6, 1)$

$f(\pi/6, 1) = \sin(\pi/6) = \frac{1}{2}$, $\quad f_u(uw) = w\cos(uw)$

$f_u(\pi/6, 1) = \cos(\pi/6) = \sqrt{3}/2$

$f_w(u,w) = u\cos(uw)$

$f_w(\pi/6, 1) = \pi/6 \cos(\pi/6) = \frac{\pi\sqrt{3}}{12}$

$$\Rightarrow z = \frac{1}{2} + \frac{\sqrt{3}}{2}(u - \pi/6) + \frac{\pi\sqrt{3}}{12}(w-1)$$

$$\boxed{= \sqrt{3}/2\, u + \frac{\pi\sqrt{3}}{12} w + \frac{1}{2} - \frac{\sqrt{3}\pi}{6}}$$

3. Find the points on the graph of $z = 3x^2 - 4y^2$ at which the vector $n = \langle 3, 2, 2 \rangle$ is normal to the tangent plane.

From p.784 the normal vector of the plane tangent to the graph at (x, y) is: $n = \langle f_x(x,y), f_y(x,y), -1 \rangle$.

So, in our case, $n = \langle 6x, -8y, -1 \rangle$.

We need $\langle 6x, -8y, -1 \rangle = \lambda \langle 3, 2, 2 \rangle \Rightarrow$

$-1 = \lambda \cdot 2 \Rightarrow \lambda = -1/2$.

So, $6x = -3/2$, $-8y = -1$ $\Rightarrow$ $x = -3/12$, $y = 1/8$ $\Rightarrow \boxed{(-1/4, 1/8)}$

Section 15.5 Solutions

1. Use that chain rule to calculate $\frac{d}{dt} f(r(t))$ for $f(x, y) = 3x - 7y$, $r(t) = \langle \cos t, \sin t \rangle$, $t = 0$.

$$\frac{d}{dt} f(r(t)) = \nabla f_{r(t)} \cdot r'(t).$$

$\nabla f = \langle 3, -7 \rangle$

$\nabla f_{r(t)} = \langle 3, -7 \rangle$, $r'(t) = \langle -\sin t, \cos t \rangle \Rightarrow r'(0) = \langle 0, 1 \rangle$

So, $\boxed{\nabla f_{r(0)} \cdot r'(0) = -7}$

2. Calculate the directional derivative of $f(x,y) = xe^{-yz}$, in the direction of $v = \langle 1,1,1 \rangle$ at $P = (1,2,0)$.

We have $D_{e_v} f(a,b) = \nabla f(a,b) \cdot e_v$.

$e_v = \dfrac{\langle 1,1,1 \rangle}{\sqrt{3}}$, $\nabla f = \langle e^{-yz}, -xze^{-yz}, -xye^{-yz} \rangle$

$\nabla f(1,2,0) = \langle 1, 0, -2 \rangle$.

So, $D_{e_v} f(a,b) = \langle 1, 0, -2 \rangle \cdot \dfrac{\langle 1,1,1 \rangle}{\sqrt{3}}$

$= \dfrac{1}{\sqrt{3}} - \dfrac{2}{\sqrt{3}} = \boxed{-1/\sqrt{3}}$

3. A bug located at $(3,9,4)$ begins walking in a straight line toward $(5,7,3)$. At what rate is the bug's temperature changing if the temperature is $T(x,y,z) = xe^{y-z}$?

The bug moves in the direction $v = \langle 5,7,3 \rangle - \langle 3,9,4 \rangle$
$= \langle 2,-2,-1 \rangle$.

$e_v = \dfrac{\langle 2,-2,-1 \rangle}{3}$

$\nabla T = \langle e^{y-z}, xe^{y-z}, -xe^{y-z} \rangle$

$\nabla T(3,9,4) = \langle e^5, 3e^5, -3e^5 \rangle$. Temp. change is $\nabla T(3,9,4) \cdot e_v =$
$\frac{2}{3}e^5 - 2e^5 + e^5 = \boxed{-e^5/3}$

Since $D_u T(p) = \nabla T_p \cdot u$

$= \|\nabla T_p\| \cdot \cos\theta$

The change in temperature is maximized if the angle between ∇T_p and u is $0 \Rightarrow$

$u = \nabla T(3,9,4) = \langle e^5, 3e^5, -3e^5 \rangle$.

For the minimum change the angle should be $\pi/2$, one such option is $u = \langle 0, 1, 1 \rangle$.

4. Let $f(x,y) = \tan^{-1}(x/y)$ and $u = \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$.

(a) Calculate ∇f.

$$\nabla f = \left\langle \frac{1}{1+x^2/y^2} \cdot \frac{1}{y}, \frac{1}{1+(x^2/y^2)} \cdot \frac{x}{y^2} \right\rangle = \left\langle \frac{y}{y^2+x^2}, -\frac{x}{y^2+x^2} \right\rangle$$

(b) Calculate $D_u f(1,1)$ and $D_u f(\sqrt{3},1)$.

$f(1,1) = \tan^{-1}(1) = \pi/4$

$f(\sqrt{3},1) = \tan^{-1}(\sqrt{3}) = \pi/3$,

$\|u\| = \sqrt{\frac{2}{4} + 2/4} = \sqrt{4/4} = 1$.

So, u is already a unit vector.

Then, $D_u f(1,1) = \nabla f_{(1,1)} \cdot \langle \sqrt{2}/2, \sqrt{2}/2 \rangle = \langle 1/2, -1/2 \rangle \cdot \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$

$= \sqrt{2}/4 - \sqrt{2}/4 \boxed{= 0}$

$D_u f(\sqrt{3},1) = \nabla f_{(\sqrt{3},1)} \cdot \langle \sqrt{2}/2, \sqrt{2}/2 \rangle = \left\langle \frac{1}{4}, -\frac{\sqrt{3}}{4} \right\rangle \cdot \langle \sqrt{2}/2, \sqrt{2}/2 \rangle$

$= \frac{\sqrt{2}}{8} - \frac{\sqrt{6}}{8} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{8}}$

(c) Show that the lines $y = mx$ for $m \neq 0$ are level curves for f.

f is not defined for $y = 0$. For $x = 0$ we have

$\tan^{-1}(0) = 0 \Rightarrow$ level curve is the y-axis.

For $y \neq 0$, $x \neq 0$, the level curves of f are the curves where $f(x,y)$ is constant. That is,

$$\tan^{-1}(x/y) = k \quad (k \neq 0) \Rightarrow$$

$$\frac{x}{y} = \tan k \Rightarrow y = \frac{1}{\tan k} x \Rightarrow$$

level curves are lines.

(d) Verify that ∇f_P is orthogonal to the level curve through P for $P = (x,y) \neq (0,0)$

Fix $(x_0, y_0) \neq (0,0)$. Then, $\frac{1}{\tan k} = \frac{y_0}{x_0}$ So, the level curve passing through this point is $y = \frac{y_0}{x_0} x$.

$\nabla f(x_0, y_0) = \left\langle \frac{y_0}{y_0^2 + x_0^2}, \frac{-x_0}{y_0^2 + x_0^2} \right\rangle$. $(0,0)$ and $(1, y_0/x_0)$ are on the line so, a direction vector for the line is $\langle 1, y_0/x_0 \rangle$.

So, $\nabla f(x_0, y_0) \cdot \langle 1, y_0/x_0 \rangle = \frac{y_0}{y_0^2 + x_0^2} - \frac{x_0 y_0}{x_0(y_0^2 + x_0^2)} = 0$. ✓

5. Find a normal vector to the surface $x^2+y^2-z^2=6$ at the point $P=(3,1,2)$.

From theorem 5 on p. 799,

$n = \nabla f_P$. $\nabla f = \langle 2x, 2y, -2z\rangle$

$\nabla f(3,1,2) = \langle 6, 2, -4\rangle$.

where $f(x,y,z) = x^2+y^2-z^2$

6. Find the two points on the ellipsoid

$\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$ where the tangent plane

where $f(x,y,z) = x^2/4 + y^2/9 + z^2$.

is normal to $v = \langle 1, 1, -2\rangle$.

We have $\nabla f = \langle x/2, 2y/9, 2z\rangle$.

So, we need $\nabla f = kv \Rightarrow$

$x = 2k, \quad y = 9k/2, \quad z = -k.$

Plugging back in ellipsoid equation,

$$\frac{4k^2}{4}+\frac{81k^2}{4\cdot 9}+k^2 = 1 \Rightarrow$$

$$k^2\left(\frac{4+9+4}{4}\right) = 1 \Rightarrow$$

$$k^2\left(\frac{101}{4}\right) = 1 \Rightarrow$$

$$k = \pm\sqrt{4/17} = \pm 2/\sqrt{17}. \quad \boxed{\text{So, } (x,y,z) = \pm\left(\frac{4}{\sqrt{17}}, \frac{9}{\sqrt{17}}, -\frac{2}{\sqrt{17}}\right)}$$